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Differential Geometry

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目 录

目录

1. Parametrized curves

Definition(Definition of differentiable curve). A parametrized differentiable curve is an infinitely differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval into $\mathbb{R}^3$.

The variable t is called the parameter of the curve. We do not exclude $a = -\infty$ and $b = \infty$.

The vector

$$\alpha'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3 \tag{1}$$

is called the **tangent vector** of the curve α at t.

For the study of the differential geometry of a curve, it is essential to assume that there exists such a tangent line at every point.

Definition(Singular point, regular curve). (i) A point $t \in I$ is called a **singular point** of α, if $\alpha'(t) = 0$.

(ii) A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be **regular** if $\alpha'(t) \neq 0$ for all $t \in I$.

A parameter called **arc length** is usually useful in further analysis.

Definition(Definition of Arc length). Given $t_0 \in I$, the **arc length** of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ from the point t_0 is defined by

$$s = \int_{\{t_0\}}^{t} |\alpha'(\tau)|\,\mathrm{d}\tau, \tag{2}$$

where $|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

Since $|\alpha'(t)| > 0$ for regular curve.

Now we talk about some invariance under reparametrization.

2. Vector product on $\mathbb{E}^3$

Definition(Equivalence: orientation). Suppose $\{e_n\}, \{f_n\}$ are two basis of n-dimensional space. They are said to have the same orientation, denoted by $e \sim f$, if the matrix of change of basis has positive determinant.

Easy to show that orientation satisfies the equivalent relationship.

The vector product of u and v (in that order) is the unique vector $u \times v \in \mathbb{R}^3$ such that

$$(u \times v) \cdot w = \det(u, v, w), \quad \forall w \in \mathbb{R}^3. \tag{3}$$

Easy to show that $(u \times v) \cdot u = 0$ and $(u \times v) \cdot v = 0$, so $u \times v \neq 0$ is orthogonal to a plane generated by u and v. To give a geometric interpretation of its norm and its direction, we proceed as follows.

Theorem(Deduction). (i) Observe that $(u \times v) \cdot (u \times v) = |u \times v|^2 > 0$, so the determinant of $(u, v, u \times v)$ is positive and it could be a basis.

(ii) Prove that

$$(u \times v) \cdot (x \times y) = \begin{vmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{vmatrix} \tag{4}$$

where u, v, x, y are arbitrary vectors. Check for basis.

and show that

$$|u \times v|^2 = |u|^2 |v|^2 (1 - \cos^2 \theta) = A^2. \tag{5}$$

(iii) The vector product is not associative. Because

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u, \tag{6}$$

and check for all basis.

3. The local theory of curves

3.1. Curvature

Let s be the arc length, and α be parametrized by s. $\alpha'(s)$ has unit length, and $|\alpha''(s)|$ measures how rapid the curve pulls away from the tangent line $\alpha'(s)$, in the neighborhood of s.

Definition(Curvature of a curve). Let $\alpha : I \to \mathbb{E}^3$ be a curve parametrized be arc length $s \in I$. The number $\kappa(s) := |\alpha''(s)|$ is called the curvature of α at s.

Easy to show that for straight line, $\alpha = us + v$, if and only if $\kappa(s) = 0$. If we have $\kappa(s) \neq 0$ unless for $s = s_0$, we could still find its curvature by leveraging the limit. If for its neighborhood, $\kappa(s) = 0$, then it is a straight line.

When change the direction, we have tangent vector changes but the curvature does not. This is because, let $\beta(s) = \alpha(-s)$, then

$$\frac{\mathrm{d}\beta(s)}{\mathrm{d}s} = \frac{\mathrm{d}\alpha(-s)}{\mathrm{d}s} = (-1)\frac{\mathrm{d}\alpha(-s)}{\mathrm{d}(-s)}. \tag{7}$$

1 ▶ Curves

If $\kappa(s) \neq 0$, then we have a unit **normal vector** n well defined by $\alpha''(s) = \kappa(s)n(s)$. The plane composed by t and n are called the **osculating plane** at s.

Example. For $\kappa(s) = 0$, check the following example.

$$\alpha(t) = \begin{cases} \left(t, 0, e^{-\frac{1}{t^2}}\right), t > 0 \\ (0,0,0), \qquad t = 0 \\ \left(t, e^{-\frac{1}{t^2}}, 0\right), t < 0. \end{cases} \tag{8}$$

To proceed with the local analysis of curves, we assume $\alpha''(s) \neq 0$ (the singular point of order 1, and $\alpha'(s) = 0$ is called the singular point of order 0).

For a plane curve, we have the following description.

Example. Assume that $\alpha(I) \subset \mathbb{E}^2$ and give κ a sign as in the text.

Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of $t(s)$ agree with the origin of $\mathbb{E}^2$; the end points of $t(s)$ then describe a parametrized curve $s \to t(s)$ called the **indicatrix** of tangents of α.

Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of $\mathbb{E}^2$. notice that we are assuming that $\kappa \neq 0$. Show that

(a) The indicatrix of tangents is a regular parametrized curve.

(b) $\mathrm{d}t/\,\mathrm{d}s = (\mathrm{d}\theta/\,\mathrm{d}s)n$, that is, $\kappa = \mathrm{d}\theta/\,\mathrm{d}s$.

Proof: (a) Easy to see since $|t'(s)| = |\kappa(s)| \neq 0$.

(b) Let

$$t(s) = [\cos\theta(s), \sin\theta(s)], \tag{9}$$

then

$$t'(s) = [-\sin\theta(s), \cos\theta(s)]\theta'(s) = \theta'(s)n, \tag{10}$$

since we have $n = jt$ (rotation by $\frac{\pi}{2}$).

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3.2. Tortion

Already we have $t' = \kappa n$. Now we do not check n', but check $b(s) := t(s) \times n(s)$, which is called **binormal vector** at s, also a unit vector representing the osculating plane. $|b'(s)|$ meausres the rate of change of the neighboring osculating plane. We claim that $b'(s)$ is parallel with $n(s)$. Indeed,

$$b' = t' \times n + t \times n' = t \times n'. \tag{11}$$

The focus is not n', but $t \perp b'$. With $b' \perp b$, we have the result.

Extract the number out and define $b'(s) = \tau(s)n(s)$ to be the following concept.

Definition(Definition of tortion). Let $\alpha : I \to \mathbb{E}^3$ be a curve parametrized be arc length $s \in I$ such that $\alpha''(s) \neq 0$. The number $\tau(s) := |b'(s)|$ is called the **tortion** of α at s.

Also, α is a plane curve, if and only if $|b'(s)| \equiv 0$. Necessarily speaking, it is easy. On the other hand, we shall show that $b(s) := b_0$, take a inner product $\alpha(s) \cdot b_0$ and check it is also a constant, which is exactly the parametrized form of plane.

Tortion could be either positive or negative.

3.3. Frenet trihedron

We have associated three orthonomal unit vector $t(s), n(s), b(s)$, which is referred to as the **Frenet trihedron** at s. And we already known that $t' = \kappa n$, and $b' = \tau n$, for $n = b \times t$ we

$$n' = b' \times t + b \times t' = -\tau b - \kappa t. \tag{12}$$

We call the above three equations the **Frenet formulas**.

Now we give the core theorem of this chapter.

Theorem(Fundamental theorem of the local theory of curves). Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exsits a regular parametrized curve $\alpha : I \to \mathbb{E}^3$ such that s is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the tortion of α.

Moreover, any other curve $\overline{\alpha}$, which satisfies the same condition, differs from α by a **rigid motion**. That is, there exists a orthonormal map ρ with positive determinant, and a translation vector c, such that $\overline{\alpha} = \rho \circ \alpha + c$.

Proof: **Proof for uniqueness.** (Details omitted)

Proof for existence. This is by ODEs theory.

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For plane curve, we could have a simpler version of the above theorem.

Example. Given a function $\kappa(s)$, show that the parametrized plane curve have κ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\, \mathrm{d}s + a, \int \sin\theta(s)\, \mathrm{d}s + b \right) \tag{13}$$

where $\theta(s) = \int \kappa(s)\,\mathrm{d}s + \varphi$. The curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

3.4. Calculations

For general regular parametrized curve $\alpha(t)$, we have the following formula for calculating the geometric variables.

Theorem(Calculations of curvature). (i) generally speaking, we have

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}. \tag{14}$$

(ii) for plane curve $\alpha'(t) = [x(t), y(t)]$, we have signed curvature

$$\kappa(t) = \frac{x'y'' - y'x''}{|x'^2 + y'^2|^{\frac{3}{2}}}. \tag{15}$$

Theorem(Calculation of tortion). Generally speaking, we have

$$\tau(t) = -\frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{|\alpha'(t) \times \alpha''(t)|^2}. \tag{16}$$

Proof:

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4. The local canonical form

Let $\alpha : I \to \mathbb{E}^3$ be a curve parametrized by arc length without singular points of order 1. We now consider the equations in a neighborhood of s_0 using the trihedron $t(s_0), n(s_0), b(s_0)$ as a basis for $\mathbb{E}^3$. We may assume without loss of generality, that $s_0 = 0$, and consider Taylor expansion

$$\alpha(s) - \alpha(0) = \alpha'(0)s + \frac{1}{2}\alpha''(0)s^2 + \frac{1}{6}\alpha'''(0)s^3 + R(s), \tag{17}$$

where $\frac{R(s)}{s^3} \to \mathbf{0}$ as $s \to 0$. Using $\alpha'(0) = t$, $\alpha''(0) = \kappa n$, and $\alpha'''(0) = (\kappa n)' = \kappa' n - \kappa^2 t - \kappa\tau b$, we rewrite the above equation sorted by t, n, b

$$\alpha(s) - \alpha(0) = \left(s - \frac{1}{6}\kappa^2 s^3\right)t + \left(\frac{1}{2}\kappa s^2 - \frac{1}{6}\kappa' s^3\right)n - \frac{1}{6}\kappa\tau s^3 b + R(s), \tag{18}$$

with

$$\begin{cases} x(s) = s - \frac{1}{6}\kappa^2 s^3 + R_x \\ y(s) = \frac{1}{2}\kappa s^2 - \frac{1}{6}\kappa' s^3 + R_y \\ z(s) = -\frac{1}{6}\kappa\tau s^3 + R_z \end{cases} \tag{19}$$

which is called the local canonical form of α.

5. Classical form of curves

Example. Given the parametrized curve

$$\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right) \tag{20}$$

where $c^2 = a^2 + b^2$. Show that

(a) s is the arc length. (i.e. $|\alpha'(s)| = 1$).

(b)

$$\kappa(s) = \frac{|a|}{c^2}, \quad \tau(s) = \frac{b}{c^2}. \tag{21}$$

Example. A curve α is called a **helix** if the tangent line of α make a constant angle with a fixed direction. Assume $\tau(s) \neq 0$, $s \in I$, show the following statements are equivalent:

(i) α is a helix,

(ii) $\kappa/\tau = \text{const}.$

(iii) the lines containing $n(s)$ and passing $\alpha(s)$ are parallel to a fixed plane.

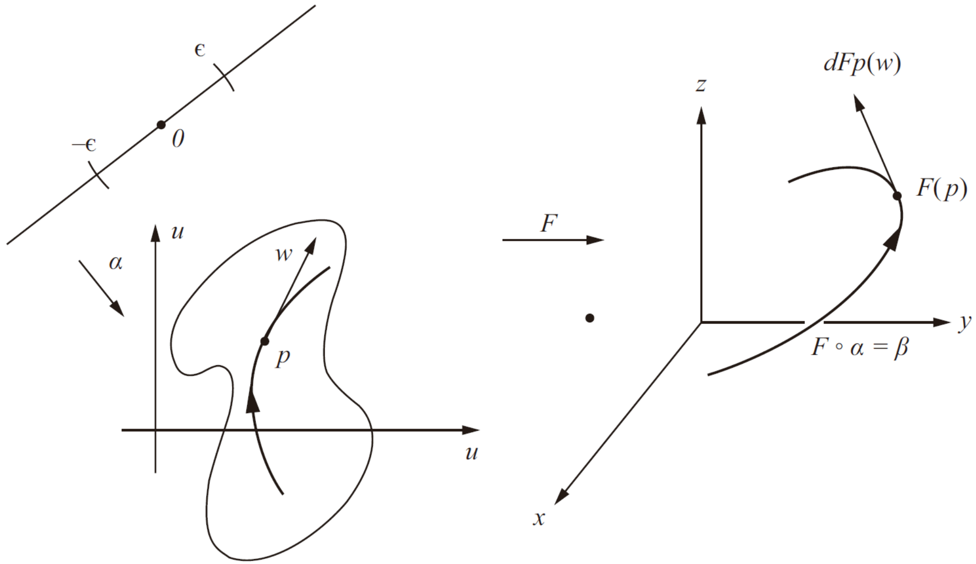
(iv) the lines containing $b(s)$ and passing $\alpha(s)$ make a constant angle with a fixed direction.

目录

1. Regular surfaces

Before stepping into the definition of regular curves, we have to define a **differential of a map**.

Definition(Definition of a differential of a map). Let $F : U \subset \mathbb{E}^n \to \mathbb{E}^m$ be a differentiable map, i.e. each component function has continuous partial derivatives w.r.t each variable. $p \in U$. A linear map $\mathrm{d}F_p : \mathbb{E}^n \to \mathbb{E}^m$ is called the **differential** of F at p, and is defined as follows.



Let $w \in \mathbb{E}^n$ and $\alpha : (-\varepsilon, +\varepsilon) \to U$ is a differentiable curve such that $\alpha(0) = p$, $\alpha'(0) = w$. Then by chain rule, $\beta = F \circ \alpha : (-\varepsilon, +\varepsilon) \to \mathbb{E}^m$ is also differentiable. Then

$$\mathrm{d}F_{p(w)} := \beta'(0). \tag{22}$$

Theorem(Differential of a map is independent of choice of curves). The above definition of $\mathrm{d}F_p$ does not depend on the choice of the curve which passes through p with tangent vector w.

Proof: We prove the case when $n = 2, m = 3$. Let $\alpha(t) = (u(t), v(t))^T$, and $F(u, v) = (x(u, v), y(u, v), z(u, v))^T$, then

$$\beta'(0) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \mathrm{d}F_{p(w)}. \tag{23}$$

which is a linear map.

Actually the above map has a matrix in canonical bases, which is usually called the **Jacobian matrix**.

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Definition(definition of regular surfaces). A subset $S \subset \mathbb{E}^3$ is a regular surface if for each $p \in S$, there exists a neighborhood V in $\mathbb{E}^3$ and a map $\boldsymbol{x} : U \to V \cap S$ of an open set $U \in \mathbb{E}^2$ onto $V \cap S \subset \mathbb{E}^3$ such that

(i) $\boldsymbol{x}$ is differentiable, i.e.

$$\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U \tag{24}$$

whose component functions have continuous partial derivatives of all orders in U.

(ii) $\boldsymbol{x}$ is a homeomorphism.

(iii) **regularity condition**. For each $q \in U$, the differential $\mathrm{d}\boldsymbol{x}_q : \mathbb{E}^2 \to \mathbb{E}^3$ is one-to-one.

Condition (i) is necessary if we want to do some geometric analysis on S. The homeomorphism in Condition (ii) prevents the self-intersections in regular surfaces, otherwise it would induce ambiguous tangent plane at the intersection point. Condition (iii) guarantee the existence of a tangent plane at all points of S. A more familiar form of condition (iii) is given as follows.

Theorem(Interpretation of condition (iii)). Let us compute the matrix of the linear map $\mathrm{d}\boldsymbol{x}_q$ in the canonical bases e_1, e_2 of $\mathbb{R}^2$ with coordinate (u,v) and f_1, f_2, f_3 of $\mathbb{R}^3$ with coordinate (x,y,z).

Let $q = (u_0, v_0)$, then $e_1 = (1,0)$ is tangent to the curve $u \mapsto (u, v_0)$ on $\mathbb{R}^2$ whose image is $u \mapsto (x(u,v_0), y(u,v_0), z(u,v_0))$ (This image curve is called the coordinate curve $v = v_0$, or with ODE $\mathrm{d}v = 0$), which lies on S and has a tangent vector at $\boldsymbol{x}_q$

$$\frac{\partial \boldsymbol{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)^T = \mathrm{d}\boldsymbol{x}_q(e_1). \tag{25}$$

Similarly, we have

$$\frac{\partial \boldsymbol{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)^T = \mathrm{d}\boldsymbol{x}_q(e_2). \tag{26}$$

So we could write the matrix of $\mathrm{d}\boldsymbol{x}_q$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}. \tag{27}$$

condition (iii) requires the matrix to be full rank. Equivalently speaking, we need $\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \neq 0$; or one of the minors of order 2 of the matrix of $\mathrm{d}\boldsymbol{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)} \tag{28}$$

does not vanish at q.

Condition (iii) is also of great importance for $\boldsymbol{x}^{-1}$ to be a so-called differentiable function, that is, if we lift its range to three dimension, then the map is differentiable. Details could be found in Change of parameters.

Actually, it would be tiresome if we test all the three conditions one by one. The following theorems gives a cheaper method to the testing by utilizing the image of a multi-variable function.

1.1. Images

Theorem(images implies regularity). If $f : U \to \mathbb{E} \in C^1(U)$ where $U \subset \mathbb{E}^2$ is an open set, then the graph of f, viewed in $\mathbb{E}^3$, i.e. the subset of $\mathbb{E}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$ is a regular surface.

Proof: Easy to show that condition (i) and (iii) are satisfied by taking derivatives and showing that $\frac{\partial(x,y)}{\partial(u,v)} = 1$. As for condition (ii), we only need to show that $\boldsymbol{x}^{-1}$ is continuous, which is obvious if we check it as a projection from $\mathbb{E}^3$ onto $\mathbb{E}^2$.

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Now we give some definitions about the following application.

Definition(regular point, critical point). Given a differentiable map $F : U \subset \mathbb{E}^n \to \mathbb{E}^m$ where U is open, a point $p \in U$ is called a **critical point** of F if the differential $\mathrm{d}F_p : \mathbb{E}^n \to \mathbb{E}^m$ is not a surjective mapping.

The image $F(p) \in \mathbb{E}^m$ of a **critical point** is called the critical value of F. A non-critical value of $\mathbb{E}^m$ is called the **regular value** of F.

The above terminology is inspired by a real-valued function of a real variable.

Example. Now particularly we consider $f : U \subset \mathbb{E}^3 \to \mathbb{E}$, which takes $m = 1, n = 3$. With a similar logic as we have in Interpretation of condition (iii), for canonical bases f_1, f_2, f_3, we have the matrix form

$$\mathrm{d}f_p = \left(f_x, f_y, f_z\right). \tag{29}$$

In this case, $\mathrm{d}f_p$ is not surjective at p, iff $f_x = f_y = f_z = 0$ at p.

From the above multi-variable function, we could find a regular surface.

Theorem(regular surfaces by images). If $f : U \subset \mathbb{E}^3 \to \mathbb{E}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{E}^3$.

Actually, this is a trick that we choose a plane in $\mathbb{E}^3$ to find a regular surface. The image of f corresponds to a image of another function h which could give an arbitrary regular surface.

Proof: Let $p = (x_0, y_0, z_0)$ be a point of $f^{-1}(a)$. Since a is a regular value of f, we may assume without loss of generality that $f_z \neq 0$ at p. Then define a map like an image

$$F(x, y, z) = (x, y, f(x, y, z))^T \tag{30}$$

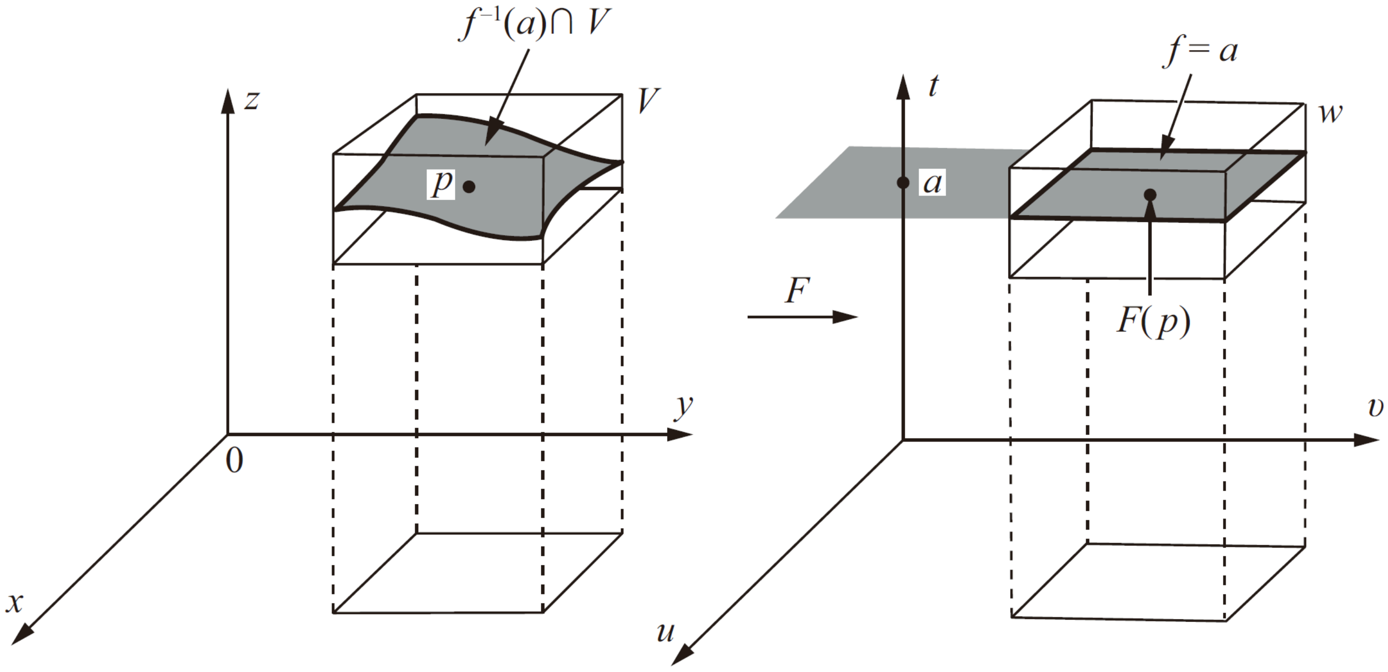
and we indicate by (u, v, t) the coordinates of a point in $\mathbb{E}^3$ where F takes its values. The matrix of the differential map $\mathrm{d}F_p$ is given by

$$\mathrm{d}F_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix} \tag{31}$$

as we illustrated in special case for function on $\mathbb{E}^3$. Whence $\det(\mathrm{d}F_P) = f_z \neq 0$. We may apply the inverse function theorem, which guarantees the existence of neighborhood V of p and W of $F(p)$ such that $F : V \to W$ is invertible, and the inverse $F^{-1} : W \to V$ is differentiable. It follows that

$$x = u, \quad y = v, \quad z = g(u, v, t), \quad (u, v, t) \in W \tag{32}$$

are differentiable. In particular, $z = g(u, v, t = a) = h(x, y)$ is differentiable defined in the projection of V onto xy plane. To use the previous proposition, we only have to show that h is differentiable.



Since

$$F(f^{-1}(a) \cap V) = \{(u, v, t) : t = a\} \cap W \tag{33}$$

we conclude that the graph of h (i.e. z) is $f^{-1}(a) \cap V$. By images implies regularity, we have $f^{-1}(a) \cap V$ is a coordinate neighborhood of p. Therefore, every point $p \in f^{-1}(a)$ can be covered by a coordinate neighborhood and $f^{-1}(a)$ is a regular surface.

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It would be good if readers could recall the implicit function theorem and its application – inverse function theorem.

The following proposition shows that any regular surface is locally the graph of a differentiable function.

Theorem(Find differentiable function using projection). Let $S \subset \mathbb{E}^3$ is a regular surface, and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function, which belongs to one of the three forms $z = f(x, y), y = g(x, z), x = h(y, z)$.

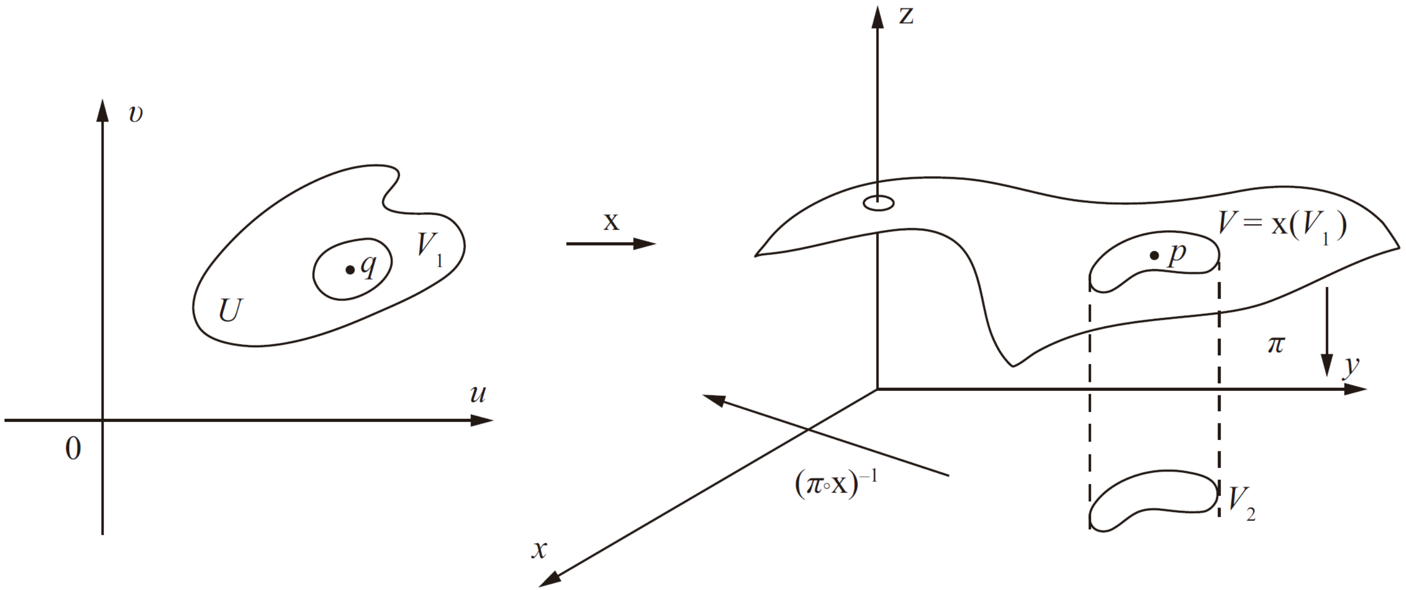
Proof: Using projection and by inverse function theorem. Without generality, we assume

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0. \tag{34}$$

We shall find an inverse function of $\pi \circ \boldsymbol{x}$, denoted by $(\pi \circ \boldsymbol{x})^{-1}$ and compose it with $z = z(u, v)$, we could have

$$z = z(u(x, y), v(x, y)) := f(x, y) \tag{35}$$

which is also differentiable.



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Using the above proposition, we claim that, for a regular surface, and any other parametrization $\boldsymbol{x}$, we do not need to test continuity of $\boldsymbol{x}^{-1}$, provided that the other conditions hold.

Example. **Example**. Show that one-sheeted cone, with its vertex at the origin, i.e.

$$S = \{(x, y, z) : z^2 = x^2 + y^2, z \geq 0\} \tag{36}$$

is not a regular surface.

Proof: The problem is at the origin. By

$$z = +\sqrt{x^2 + y^2} \tag{37}$$

is not differentiable at $(0, 0)$.

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Theorem(omit the test of continuity of inverse map). Let $p \in S$ be a point of a regular surface S and $\boldsymbol{x} : U \subset \mathbb{E}^2 \to \mathbb{E}^3$ is a parametrization with $p \in \boldsymbol{x}(U)$ such that condition 1 and 3 of definition of regular surfaces hold. Assume $\boldsymbol{x}$ is one-to-one, then $\boldsymbol{x}^{-1}$ is continuous.

2. Change of parameters

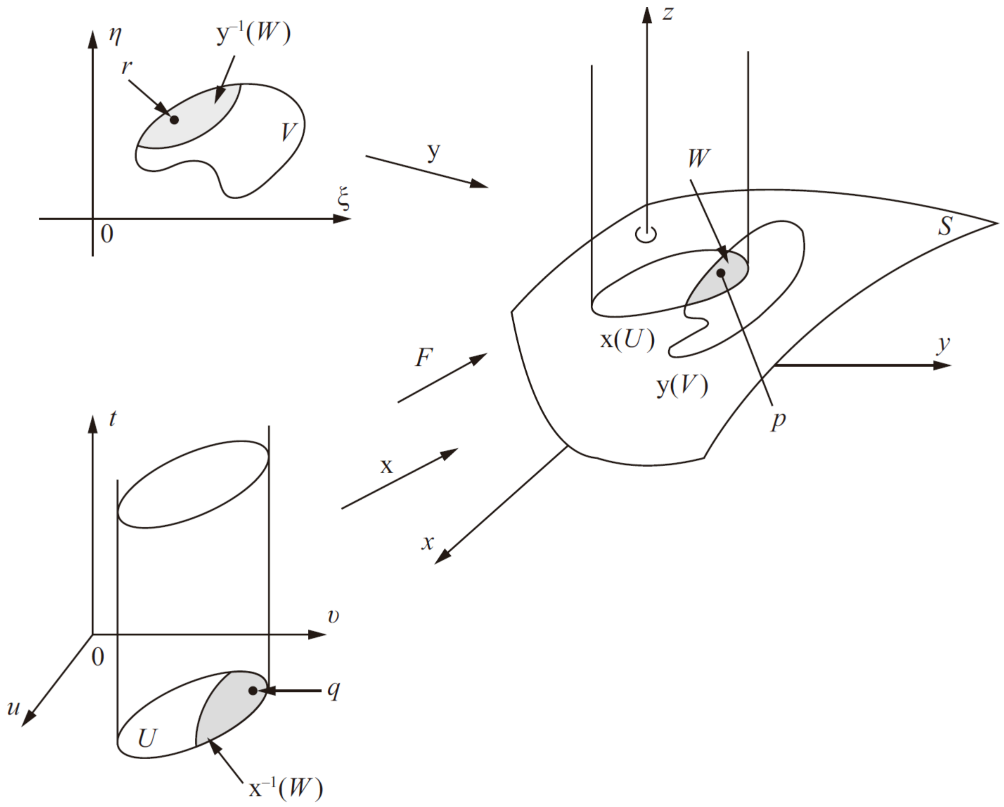
Theorem(Differentiability of change of parameters). Let p be a point in a regular surface $S \subset \mathbb{E}^3$, and two parametrizations $\boldsymbol{x} : U \subset \mathbb{E}^2 \to \mathbb{E}^3$ and $\boldsymbol{y} : V \subset \mathbb{E}^2 \to \mathbb{E}^3$, parametrized by (u, v) and (ξ, η), respectively. Suppose $p \in \boldsymbol{x}(U) \cap \boldsymbol{y}(V) = W$. Then the change of parameters $h = \boldsymbol{x}^{-1} \circ \boldsymbol{y} : \boldsymbol{y}^{-1}(W) \to \boldsymbol{x}^{-1}(W)$ is a diffeomorphism.

Proof: Utilizing the map

$$F(u, v, t) \to (x(u, v), y(u, v), z(u, v) + t), \quad (u, v) \in U \tag{38}$$

is a diffeomorphism by condition (iii). Restrict the map on a slice $U \times \{0\}$ and F^{-1} is differentiable.

Check the following figure.



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The definition of differentiability could be extended to mappings between surfaces by utilizing the differentiability of maps between plane parameters.

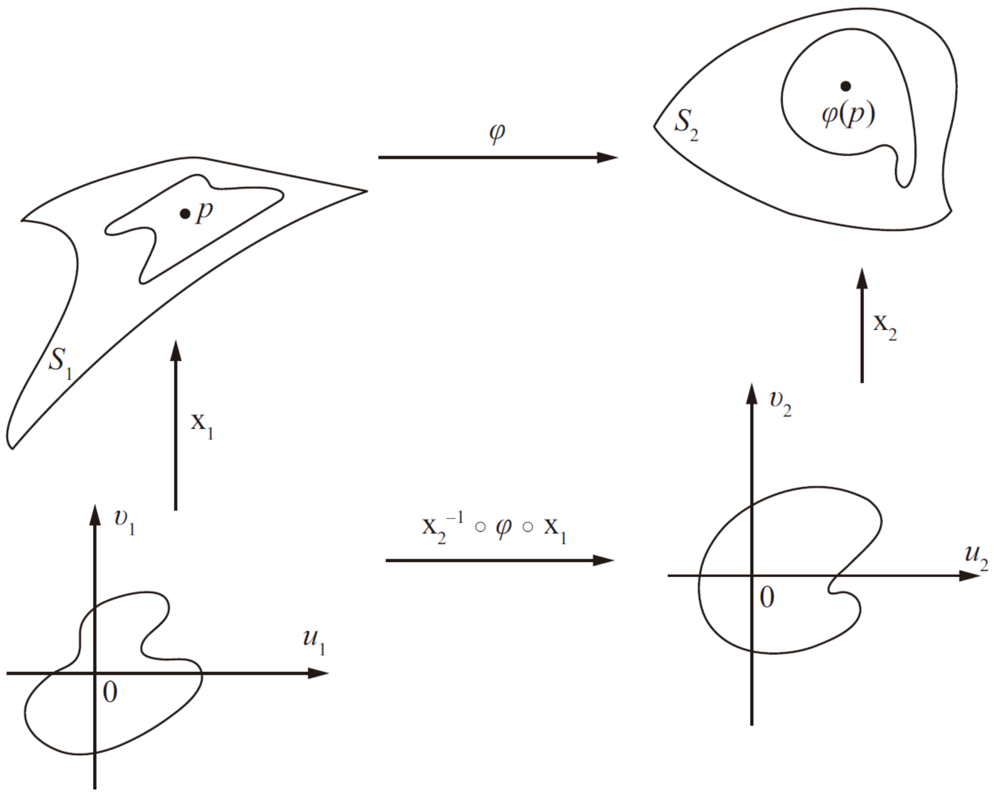
Definition(differentiability of maps between surfaces). A continuous map $\varphi : V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2, is said to be **differentiable** at $p \in V_1$, if for given parametrization

$$\boldsymbol{x}_1 : U_1 \subset \mathbb{E}^2 \to S_1, \quad \boldsymbol{x}_2 : U_2 \subset \mathbb{E}^2 \to S_2, \tag{39}$$

with $p \in \boldsymbol{x}_1(U_1)$ and $\varphi(\boldsymbol{x}_1(U_1)) \subset \boldsymbol{x}_2(U_2)$, the map composition

$$\boldsymbol{x}_2^{-1} \circ \varphi \circ \boldsymbol{x}_1 : U_1 \to U_2 \tag{40}$$

is differentiable at $q = \boldsymbol{x}_1^{-1}(p)$.



Two surfaces S_1 and S_2 are **diffeomorphic**, if there exists a differentiable map $\varphi : S_1 \to S_2$ with a differentiable inverse $\varphi^{-1} S_2 \to S_1$. Such a map φ is called a **diffeomorphism** between S_1 and S_2.

Example. Example. If $\boldsymbol{x} : U \subset \mathbb{E}^2 \to S$ is a parametrization, then $\boldsymbol{x}^{-1} : \boldsymbol{x}(U) \to \mathbb{E}^2$ is differentiable. This means **every regular surface is locally diffeomorphic to a plane**. This is useful in manifold learning.

Proof: Just check the differentiability of the map $I \circ \boldsymbol{x}^{-1} \circ \boldsymbol{y}$ for any two given parametrizations.

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Example. Example. Let S_1 and S_2 be regular surfaces. Assume that $S_1 \subset V \subset \mathbb{E}^3$, V is an open set of $\mathbb{E}^3$. Suppose $\varphi : V \to \mathbb{E}^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$. Then the restriction $\varphi_{|_{S_1}} : S_1 \to S_2$ is a differentiable map. The followings are some applications.

(i) **Symmetry**. S is a symmetric surface relative to xy plane. Then the differentiable map $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ defined by

$$\sigma(x, y, z) = (x, y, -z) \tag{41}$$

is differentiable restricted on S.

(ii) **Rotations**. S is a regular surface invariant by rotation $R_{z,\theta}$, which denotes a rotation of angle θ about z axis. Then the restriction

$$R_{z,\theta} : S \to S \tag{42}$$

is differentiable.

(iii) **Stretching operation**. Let $\varphi : \mathbb{E}^3 \to \mathbb{E}^3$ is a stretching map given by

$$\varphi(x, y, z) = (ax, by, cz), \quad a, b, c \neq 0. \tag{43}$$

Then $\varphi : S^2 \to$ ellipsoid

$$\left\{(x, y, z) \in \mathbb{E}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\} \tag{44}$$

is differentiable.

Proof:

■

Now we could define a regular curve using concept of maps.

Definition(definition of regular curve). A **regular curve** in $\mathbb{E}^3$ is a subset $C \subset \mathbb{E}^3$ with the following properties. For each $p \in C$, there exists a neighborhood $V \subset \mathbb{E}^3$ of p and a differentiable map $\alpha : I \subset \mathbb{E} \to C \cap V$ such that the differential $\mathrm{d}\alpha_t$ is one-to-one for each $t \in I$.

It is of the same logic to show that change of parameters of curves is given by a diffeomorphism.

By change of parameters, we could find properties independent of parameters, that is, the geometric properties.

3. The tangent plane

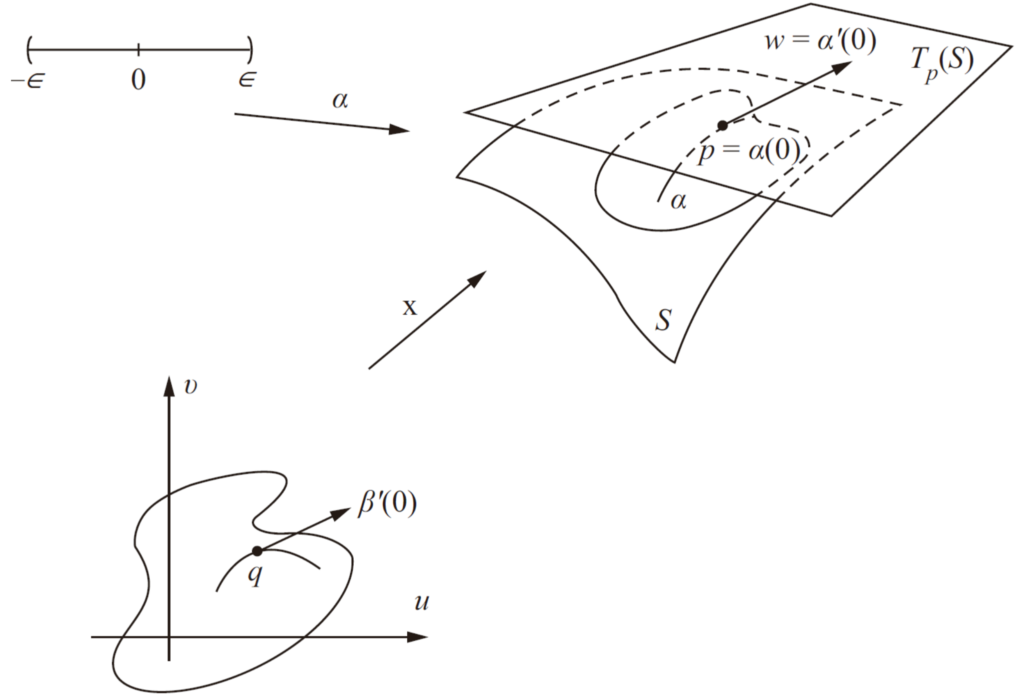
Note in the following, q is at the plane, p is on the regular surface in $\mathbb{E}^2$, and w is a velocity vector of a regular surface at p.

Theorem(Two ways of viewing tangent plane, definition of tangent plane is independent of parameters). Let $\boldsymbol{x} : U \subset \mathbb{E}^2 \to S$ be a parametrization of a regular surface, $q \in U$ is on the plane. The vector subspace of dimension 2,

$$\mathrm{d}\boldsymbol{x}_q(\mathbb{E}^2) \subset \mathbb{E}^3 \tag{45}$$

coincides with the set of tangent vectors to S at $\boldsymbol{x}_q$.

Proof: Let w be a tangent vector at $\boldsymbol{x}(q)$, i.e. $w = \alpha'(0)$, where $\alpha : (-\varepsilon, +\varepsilon) \to \boldsymbol{x}(U) \subset S$ is differentiable and $\alpha(0) = \boldsymbol{x}(q)$. By differentiability of parametrization, we have the composition $\beta = \boldsymbol{x}^{-1} \circ \alpha : (-\varepsilon, +\varepsilon) \to U$ is differentiable. Take a differential, and we have $\mathrm{d}\boldsymbol{x}_q(\beta'(0)) = w$, so $w \in \mathrm{d}\boldsymbol{x}_q(\mathbb{E}^2)$.



On the other hand, let $w = \mathrm{d}\boldsymbol{x}_q(v)$, where $v \in \mathbb{E}^2$, which is the velocity vector of the curve $\gamma : (-\varepsilon, +\varepsilon) \to U$ given by

$$\gamma(t) = tv + q, \quad t \in (-\varepsilon, +\varepsilon). \tag{46}$$

by Definition of a differential of a map, we have $\mathrm{d}\boldsymbol{x}_{p(v)} = \alpha'(0)$, with $\alpha = \boldsymbol{x} \circ \gamma$.

■

From the above definition, the plane $\mathrm{d}\boldsymbol{x}_q(\mathbb{E}^2)$ which passes $\boldsymbol{x}(q) = p$, does not depend on the parametrization $\boldsymbol{x}$. This plane is called the **tangent plane** to S at p, denoted by $T_p(S)$.

Write its bases as follows

$$\boldsymbol{x}_u := \frac{\partial \boldsymbol{x}}{\partial u}, \quad \boldsymbol{x}_v := \frac{\partial \boldsymbol{x}}{\partial v}. \tag{47}$$

Then the parametrization of vector $w \in T_{p(S)}$ are determined by

$$w = \alpha'(0) = \frac{\mathrm{d}(\boldsymbol{x} \circ \beta)}{\mathrm{d}t} = \boldsymbol{x}_u(q)u'(0) + \boldsymbol{x}_{v(q)}v'(0). \tag{48}$$

Theorem(Differential of a map between surfaces). Let S_1, S_2 be two regular surfaces and $\varphi : V \in S_1 \to S_2$ is a differentiable map of an open set V of S_1 into S_2. Given tangent vector $w = \alpha'(0) \in T_{p(S_1)}$, let $\beta : \varphi \circ \alpha$ with $\beta(0) = \varphi(p)$. Then $\beta'(0)$ does not depend on the choice of α. The map $\mathrm{d}\varphi_p : T_{p(S_1)} \to T_{\varphi(p)}(S_2)$ defined by $\mathrm{d}\varphi_{p(w)} = \beta'(0)$ is linear.

Proof: Take the partial derivatives and the proof is similar as we have in Differential of a map is independent of choice of curves.

■

4. The first fundamental form

Definition(Definition of the first fundamental form of surfaces). The first fundamental form of the regular surface S at point $p \in S$ is defined by $I_p : T_{P(S)} \to \mathbb{R}$

$$I_{p(w)} = \langle w, w \rangle = |w|^2 \geq 0. \tag{49}$$

For a parametrization of I_p, assume we have $\boldsymbol{x}(u, v)$, $\alpha : (-\varepsilon, +\varepsilon) \to S$, with $\alpha(0) = p$, with $\alpha'(0) = w$, then

$$\begin{aligned} I_{p(w)} &= \langle w, w \rangle \\ &= \langle \boldsymbol{x}_u u' + \boldsymbol{x}_v v', \boldsymbol{x}_u u' + \boldsymbol{x}_v v' \rangle \\ &= |\boldsymbol{x}_u|^2 (u')^2 + \boldsymbol{x}_u \cdot \boldsymbol{x}_v u' v' + |\boldsymbol{x}_v|^2 (v')^2 \\ &:= E(u')^2 + 2Fu'v' + G(v')^2. \end{aligned} \tag{50}$$

Now we could give some typical examples.

Example. **Example**. Calculate the first fundamental form of the regular surfaces.

(i) **Plane**. A plane that passes through $p = (x_0, y_0, z_0)$ and contain $w_1 = (a_1, a_2, a_3)$ and $b_2 = (b_1, b_2, b_3)$, is given by

$$\boldsymbol{x}(u, v) = p + uw_1 + vw_2. \tag{51}$$

where $U = \mathbb{R}^2$.

(ii) The **cylinder** over the circle $x^2 + y^2 = 1$, is given by

$$\boldsymbol{x}(u, v) = (\cos u, \sin u, v). \tag{52}$$

where $U = \{(u, v) : u \in (0, 2\pi), v \in \mathbb{R}\}$.

(iii) **Helicoid** generated by a helix $(\cos u, \sin u, au)$ given by

$$\boldsymbol{x}(u, v) = (v \cos u, v \sin u, au) \tag{53}$$

where $u \in (0, 2\pi), v \in \mathbb{R}$.

(iv) **Sphere**. A sphere given by

$$\boldsymbol{x}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u) \tag{54}$$

where $u \in (0, 2\pi), v \in (0, \pi)$.

Practically speaking, if we know I_p, then we could calculate some geometric quantity.

Theorem(calculations of geometric quantity on a regular surface). (i) **arc length**.

$$s(t) = \int_0^t |\alpha'(\tau)|\,\mathrm{d}\tau = \int_0^t \sqrt{|\alpha'(\tau)|^2}\,\mathrm{d}\tau = \int_0^t \sqrt{I(\alpha'(\tau))}\,\mathrm{d}\tau \tag{55}$$

(ii) **vector angle**.

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0)\rangle}{|\alpha'(t_0)||\beta'(t_0)|}). \tag{56}$$

(iii) **Area**. Let $R \in S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $\boldsymbol{x} : U \subset \mathbb{E}^2 \to S$. The positive number

$$A(R) := \int_Q |\boldsymbol{x}_u \times \boldsymbol{x}_v|\,\mathrm{d}u\,\mathrm{d}v, \quad Q = \boldsymbol{x}^{-1}(R). \tag{57}$$

is called the area of R. In actual calculations, we have

$$|\boldsymbol{x}_u \times \boldsymbol{x}_v| = \sqrt{|\boldsymbol{x}_u|^2|\boldsymbol{x}_v|^2 - |\boldsymbol{x}_u \cdot \boldsymbol{x}_v|^2} = \sqrt{EG - F^2}. \tag{58}$$

目录

Gauss map

Given a parametrization $\boldsymbol{x} : U \subset \mathbb{E}^2 \to S$ of a regular surface at a point $p \in S$, we could choose a unit normal vector at each point of $\boldsymbol{x}(U)$ by

$$N(p) = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{|\boldsymbol{x}_u||\boldsymbol{x}_v|}(p), \quad p \in \boldsymbol{x}(U). \tag{59}$$

Thus we have a differentiable map $N : U \subset \mathbb{E}^2 \to \mathbb{R}^3$. More generally, if $V \in S$ is an open set in S and $N : V \to \mathbb{R}^3$ is a differentiable map which associates to each $p \in V$ a unit normal vector at p, we say that N is a **differentiable field of unit normal vectors** on V.

Not all surfaces admit a differentiable field of unit vectors defined on the whole surface. For instance, the Mobius strip.

Definition(definition of Gauss map). Let $S \subset \mathbb{E}^3$ be a regular surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values at the unit sphere

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \tag{60}$$

thus $N : S \to S^2$ is called the **Gauss map** of S.

Gauss map is differentiable. The differential $\mathrm{d}N_p$ of N at $p \in S$ is a linear map from $T_{p(S)}$ to $T_{\{N(p)\}}(S^2)$. Since the two are the same space, $\mathrm{d}N_p$ could be looked upon as a linear map on $T_{p(S)}$.

Example. **Example**. check the differential of N of each surfaces.

(i) **Plane**. Norm vector is a constant, so $\mathrm{d}N_p \equiv 0$.

(ii) **Unit Sphere**. Norm vector $N = (x, y, z)$ and $\mathrm{d}N_p(v) = v$.

(iii) **Cylinder**, i.e. $x^2 + y^2 = 1$. Norm vector $N = (x, y, 0)$, and

$$\mathrm{d}N_{p(v)} = \begin{cases} \theta, v = (0, 0, z) \\ v, v = (x, y, 0). \end{cases} \tag{61}$$

Proof: Considering a curve in the surface.

■

The following is a fact about the differential of Gauss map.

Theorem(Self-adjoint map of the differential map of Gauss map). The differential $\mathrm{d}N_p : T_p(S) \to T_{\{p\}}(S)$ of the Gauss map at point $p \in S$ is a self-adjoint linear map.

Proof: We shall show that $\langle \mathrm{d}N_p(w_1), w_2\rangle = \langle w_1, \mathrm{d}N_p(w_2)\rangle$.

Now we assume $\boldsymbol{x}(u,v)$ be a parametrization of S at p, $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is the associated basis of $T_{p(S)}$. If $\alpha(t) = \boldsymbol{x}(u(t), v(t))$ is a curve in S, with $\alpha(0) = p$, then

$$\mathrm{d}N_p(\alpha'(0)) = N_u u'(0) + N_v v'(0). \tag{62}$$

where $N_u = \mathrm{d}N_{p(\boldsymbol{x}_u)}$ for u line, and $N_v = \mathrm{d}N_p(\boldsymbol{x}_v)$ for v line. So we only need to show that $\langle \mathrm{d}N_p(\boldsymbol{x}_u), \boldsymbol{x}_v\rangle = \langle \boldsymbol{x}_u, \mathrm{d}N_p(\boldsymbol{x}_v)\rangle$.

Notice that $\langle N, \boldsymbol{x}_u\rangle = 0$, so taking its derivatives w.r.t v gives

$$\langle N_v, \boldsymbol{x}_u\rangle + \langle N, \boldsymbol{x}_{uv}\rangle = 0. \tag{63}$$

taking derivatives of $\langle N, \boldsymbol{x}_v\rangle = 0$ w.r.t u gives $\langle N_u, \boldsymbol{x}_v\rangle + \langle N, \boldsymbol{x}_{vu}\rangle = 0$.

And we are done.

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Given the above fact, we could associate to $\mathrm{d}N_p$ a quadratic form Q in $T_{p(S)}$, namely $Q(v) = \langle \mathrm{d}N_{p(v)}, v\rangle$ (according to bilinear form $B(v,w) = \langle \mathrm{d}N_{p(v)}, w\rangle$ and $Q(v) = B(v,v)$).

Definition(Second fundamental form). The quadratic form $\mathbb{I}_p$, defined in $T_{p(S)}$ by

$$\mathbb{I}_p(v) = -\langle \mathrm{d}N_{p(v)}, v\rangle \tag{64}$$

is called the second fundamental form of S at p.

We give a geometric interpretation of the above second fundamental form using the **normal curvature**.

Definition(Definition of Normal curvature). Let C be a regular curve in S passing through $p \in S$, κ the curvature of C at p, with $\cos\theta = \langle n, N\rangle$ where n is the normal vector to C and N is the normal vector to S at p. Then the number $\kappa_n = \kappa\cos\theta$ is called the **normal curvature** of $C \subset S$ at p.

Theorem(Meusnier: geometric interpretation of the second fundamental form). All curves lying on S and having at a given point $p \in S$ the same tangent vector share the same normal curvature.

The above proposition allows us to speak of the normal curvature along a given direction at p.

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Thank You For Listening!